Definition. Let $A \subset \mathbb{R}$, $A^c = \mathbb{R} \setminus A$ be its complement. A point $a \in \mathbb{R}$ is called

(1) *an interior point of* A if for some $\varepsilon > 0$, $V_\varepsilon(a) \subset A$. Equivalently, $V_\varepsilon(a) \cap A^c = \emptyset$;

(2) *an exterior point of* A if for some $\varepsilon > 0$, $V_\varepsilon(a) \subset A^c$. Equivalently, $V_\varepsilon(a) \cap A = \emptyset$;

(3) *a boundary point of* A if it is neither interior nor exterior point of A. Equivalently, for all $\varepsilon > 0$, $V_\varepsilon(a) \cap A \neq \emptyset$ and $V_\varepsilon(a) \cap A^c \neq \emptyset$.

The set of all interior points of A is denoted by $\operatorname{Int} A$ (interior of A), the set of all exterior points of A is denoted by $\operatorname{Ext} A$ (exterior of A), and the set of all boundary points of A is denoted by $\operatorname{Bd} A$ (boundary of A).

Remark. (1) For any $A \subset \mathbb{R}$, $\mathbb{R} = \operatorname{Int} A \bigcup \operatorname{Bd} A \bigcup \operatorname{Ext} A$, and no two of these three sets have a non-empty intersection.

(2) It immediately follows from the definitions that $\operatorname{Int} A = \operatorname{Ext} A^c$.

(3) By the symetry of the definition of the boundary points, $\operatorname{Bd} A = \operatorname{Bd} A^c$.

(4) A is open if and only if any point of A is interior, i.e. $A = \operatorname{Int} A$.

(5) If a is an isolated point of A or A^c then $a \in \operatorname{Bd} A$.

Theorem. $\operatorname{Int} A$ *and* $\operatorname{Ext} A$ *are open sets. If* U *is open,* $U \subset A$*, then* $U \subset \operatorname{Int} A$.

Proof. Let us first establish the second statement. If U is open and $a \in U$ then $\exists \varepsilon > 0$ for which $V_\varepsilon(a) \subset U \subset A$. So a is an interior point of A, $a \in \operatorname{Int} A$. Thus $U \subset \operatorname{Int} A$.

Now let $a \in \operatorname{Int} A$. Then $\exists \varepsilon > 0$ such that $V_\varepsilon(a) \subset A$. But $V_\varepsilon(a)$ is open and contained in A, so, by what we just proved, $V_\varepsilon(a) \subset \operatorname{Int} A$. So every point of $\operatorname{Int} A$ is an interior point, and so $\operatorname{Int} A$ is open.

Since $\operatorname{Ext} A = \operatorname{Int} A^c$, it is also open. □

Theorem. $A \subset \mathbb{R}$ *is closed if and only if* $\operatorname{Bd} A \subset A$*, i. e. if and only if it contains all of its boundary points.*

Proof. Assume that A is closed and $a \in \operatorname{Bd} A$. If $a \notin A$, then, since $V_\varepsilon(a) \bigcup A \neq \emptyset$ for any $\varepsilon > 0$, we have

$$(V_\varepsilon(a) \setminus \{a\}) \bigcup A \neq \emptyset,$$

so a is a limit point of A. Since A is closed, $a \in A$ – contradiction! Thus $\operatorname{Bd} A \subset A$.

Assume now that $\operatorname{Bd} A \subset A$ and a is a limit point of A. If $a \notin A$, then

$$\forall \varepsilon > 0 \,:\, V_\varepsilon(a) \bigcup A \neq \emptyset \text{ since } a \text{ is a limit point of} A;\; V_\varepsilon(a) \bigcup A^c \neq \emptyset \text{ since } a \in A^c.$$

Thus a is a boundary point of A, so $a \in A$ – contradiction! Thus A contains all of its limit points, so it is closed. □

Theorem. A *is open if and only if* A^c *is closed.*

Proof. A is open if and only if A is equal to $\operatorname{Int} A$. This happens if and only if $\operatorname{Bd} Z = \operatorname{Bd} A^c \subset A^c$, i. .e when A^c is closed. □

Corollary. $\operatorname{Bd} A = \mathbb{R} \setminus (\operatorname{Int} A \bigcup \operatorname{Ext} A)$ *and* $A \bigcup \operatorname{Bd} A = \mathbb{R} \setminus \operatorname{Ext} A$ *are closed, as the complements of open sets.*

Definition. The *closure* of A is the set A together with all of its boundary points:

$$\overline{A} = A \bigcup \operatorname{Bd} A$$

Remark. A set A is closed if and only if it contains all of its boundary points. Since any set contains all of its interior points, A is closed if and only if $A = \overline{A}$.

Lemma. *Let F be closed, $A \subset F$. Then $\overline{A} \subset F$.*

Proof.

$$A \subset F \implies \operatorname{Ext} F \subset \operatorname{Ext} A \implies F = \overline{F} = (\mathbb{R} \setminus \operatorname{Ext} F) \supset (\mathbb{R} \setminus \operatorname{Ext} A) = \overline{A}$$

□

Lemma (Alternative characterizations of closure)**.** *(1) $a \in \overline{A}$ if and only if*

$$a = \lim_{n\to\infty} a_n \text{ with } a_n \in A.$$

(2) Let

$$\operatorname{dist}(a, A) := \inf \{|x - a| \,:\, x \in A\}$$

Then $a \in \overline{A}$ if and only if $\operatorname{dist}(a, A) = 0$.

(3) Let $L(A)$ denote the set of all limit points of A. Then $\overline{A} = A \bigcup L(A)$.